Chapter 4: Dynamic Programming

Objectives of this chapter:

- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Show how DP can be used to compute value functions, and hence, optimal policies
- Discuss efficiency and utility of DP

Policy Evaluation

Policy Evaluation: for a given policy π, compute the state-value function V^{π}

Recall: **State - value function for policy** π **:**

$$V^{\pi}(s) = E_{\pi}\left\{R_t \mid s_t = s\right\} = E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid s_t = s\right\}$$

Bellman equation for V^{π} :

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} P_{ss'}^{a}\left[R_{ss'}^{a} + \gamma V^{\pi}(s')\right]$$

— a system of $|S|$ simultaneous linear equations

Iterative Methods

$$V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_k \rightarrow V_{k+1} \rightarrow \cdots \rightarrow V^\pi$$

a "sweep"

A sweep consists of applying a **backup operation** to each state.

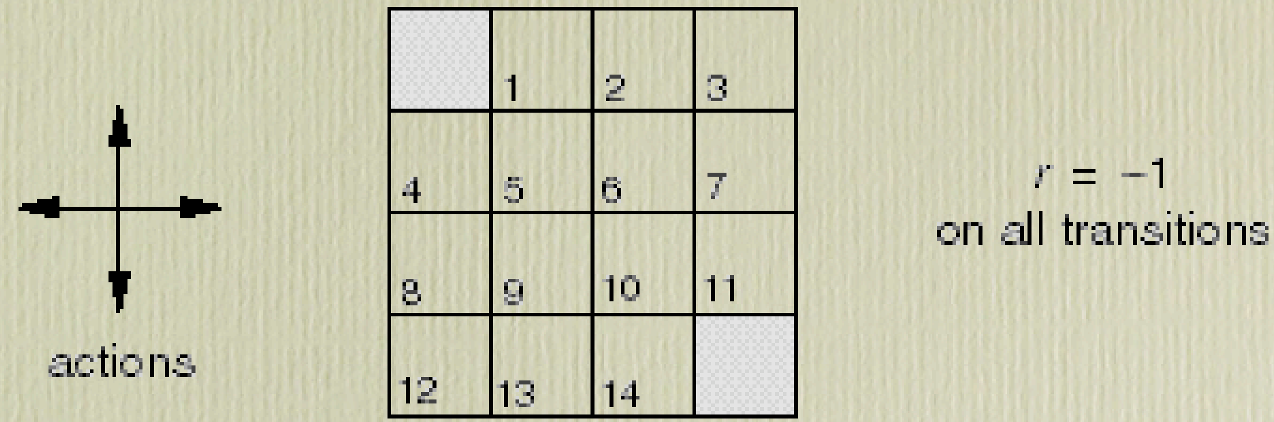
A **full policy evaluation backup**:

$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} P_{ss'}^a \left[R_{ss'}^a + \gamma V_k(s') \right]$$

Iterative Policy Evaluation

Input π, the policy to be evaluated
Initialize $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx V^\pi$

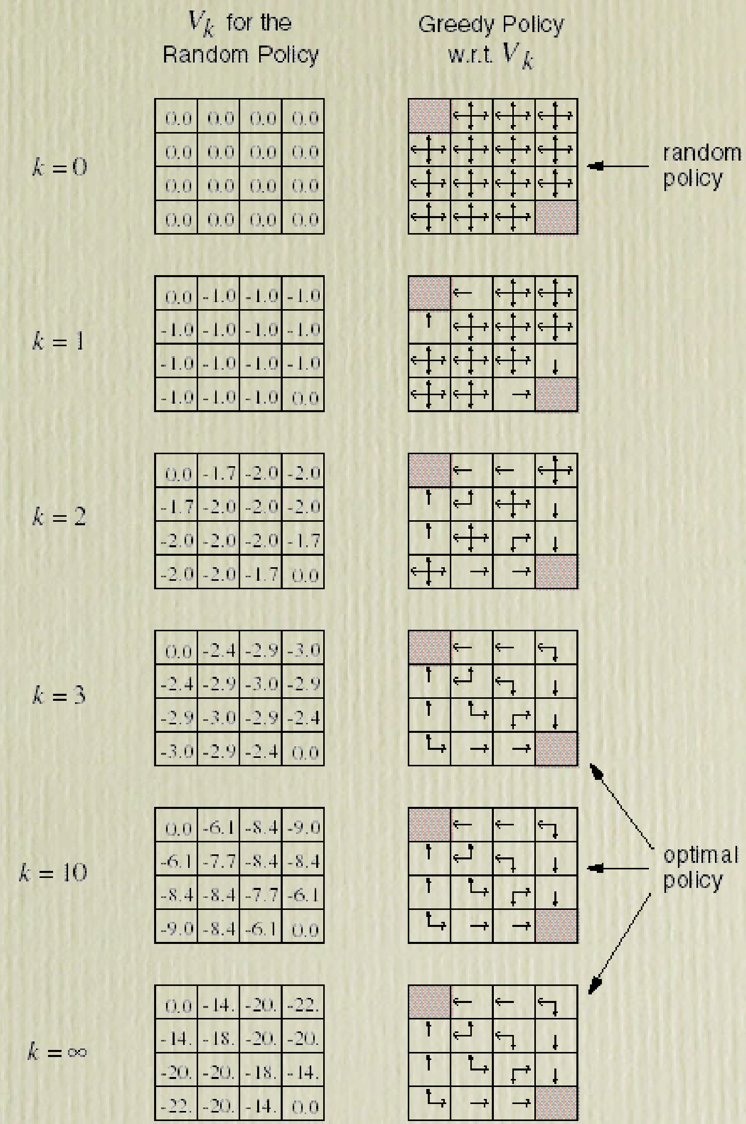
A Small Gridworld



- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is −1 until the terminal state is reached

Iterative Policy Eval for the Small Gridworld

π = random (uniform) action choices



Policy Improvement

Suppose we have computed V^{π} for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

The value of doing a in state s is:

$$Q^{\pi}(s,a) = E_{\pi}\left\{r_{t+1} + \gamma V^{\pi}(s_{t+1}) \middle| s_t = s, a_t = a\right\}$$
$$= \sum_{s'} P_{ss'}^{a}\left[R_{ss'}^{a} + \gamma V^{\pi}(s')\right]$$

It is better to switch to action a for state s if and only if

$$Q^{\pi}(s,a) > V^{\pi}(s)$$

Policy Improvement Cont.

Do this for all states to get a new policy π' that is **greedy** with respect to V^{π} :

$$\begin{aligned}\pi'(s) &= \arg\max_a Q^{\pi}(s,a) \\ &= \arg\max_a \sum_{s'} P^a_{ss'}\left[R^a_{ss'} + \gamma V^{\pi}(s')\right]\end{aligned}$$

Then $V^{\pi'} \geq V^{\pi}$

Policy Improvement Cont.

What if $V^{\pi'} = V^{\pi}$?

i.e., for all $s \in S$, $V^{\pi'}(s) = \max_a \sum_{s'} P^a_{ss'}\left[R^a_{ss'} + \gamma V^{\pi}(s')\right]$?

But this is the Bellman Optimality Equation.

So $V^{\pi'} = V^*$ and both π and π' are optimal policies.

Policy Iteration

$$\pi_0 \rightarrow V^{\pi_0} \rightarrow \pi_1 \rightarrow V^{\pi_1} \rightarrow \cdots \pi^* \rightarrow V^* \rightarrow \pi^*$$

policy evaluation

policy improvement
"greedification"

Policy Iteration

1. Initialization
   $V(s) \in \Re$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

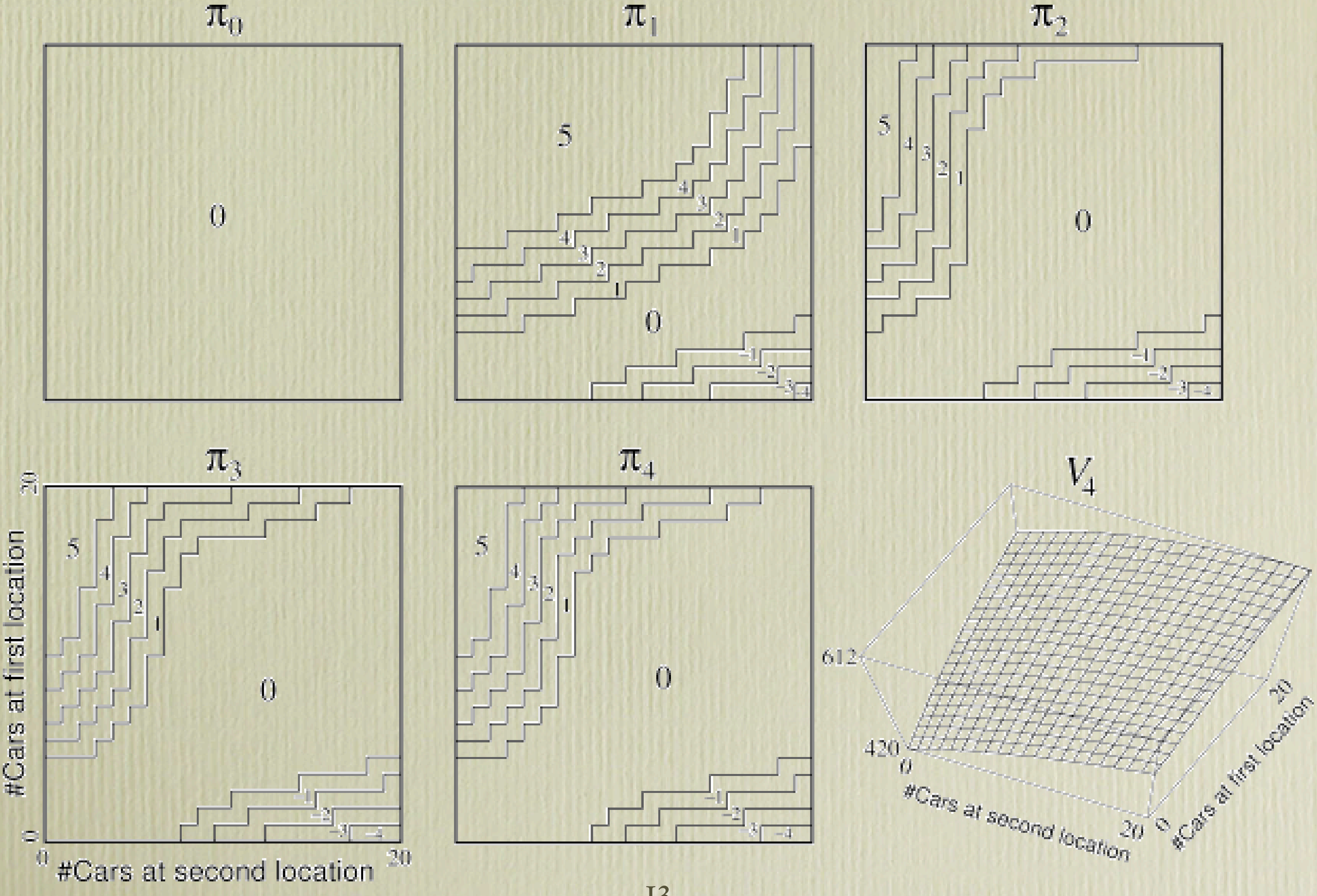
2. Policy Evaluation
   Repeat
   $\Delta \leftarrow 0$
   For each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s'} \mathcal{P}_{ss'}^{\pi(s)} \left[\mathcal{R}_{ss'}^{\pi(s)} + \gamma V(s') \right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $b \leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s'} \mathcal{P}_{ss'}^{a} \left[\mathcal{R}_{ss'}^{a} + \gamma V(s') \right]$
   If $b \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop; else go to 2

Jack's Car Rental

- $10 for each car rented (must be available when request received
- Two locations, maximum of 20 cars at each
- Cars returned and requested randomly
  - Poisson distribution, *n* returns/requests with prob $\frac{\lambda}{n!}e^{-\lambda}$
  - 1st location: average requests = 3, average returns = 2
  - 2nd location: average requests = 4, average returns = 2
- Can move up to 5 cars between locations overnight
- States, Actions, Rewards?
- Transition probabilities?

Jack's Car Rental



Jack's CR Exercise

- Suppose the first care moved is free
  - From 1st to 2nd location
  - Because an employee travels that way anyway (by bus)
- Suppose only 10 cars can be parked for free at each location
  - More than 10 cost $4 for using an extra parking lot
- Such arbitrary nonlinearities are common in real problems

Value Iteration

Recall the **full policy evaluation backup**:

$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} P_{ss'}^a \left[R_{ss'}^a + \gamma V_k(s') \right]$$

Here is the **full value iteration backup**:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P_{ss'}^a \left[R_{ss'}^a + \gamma V_k(s') \right]$$

Value Iteration Cont.

Initialize V arbitrarily, e.g.,$V(s) = 0$, for all $s \in \mathcal{S}^+$

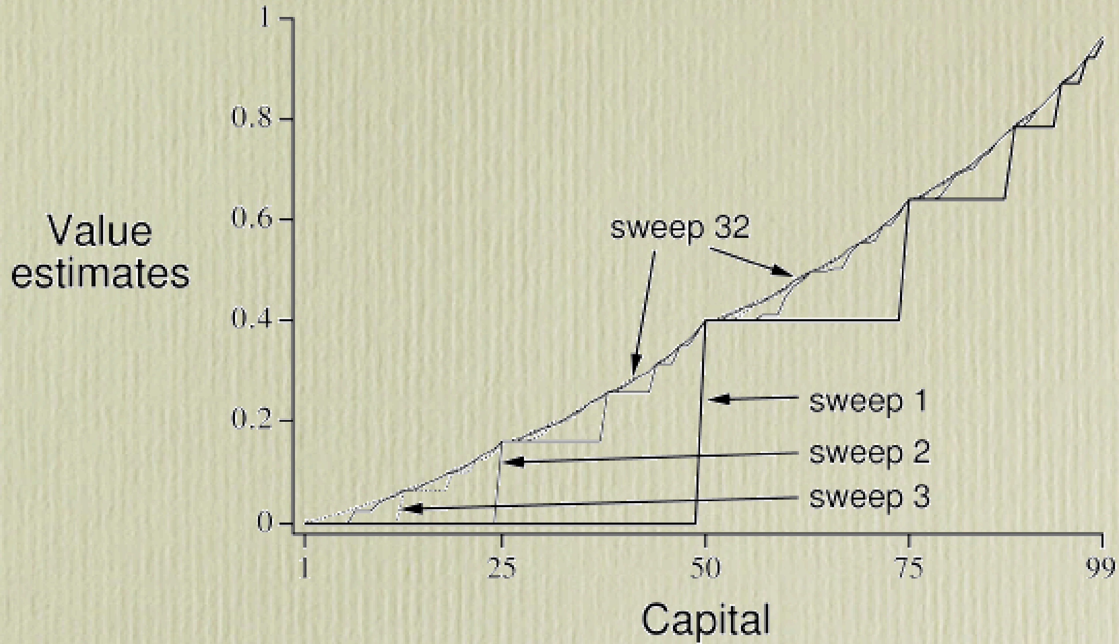
Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

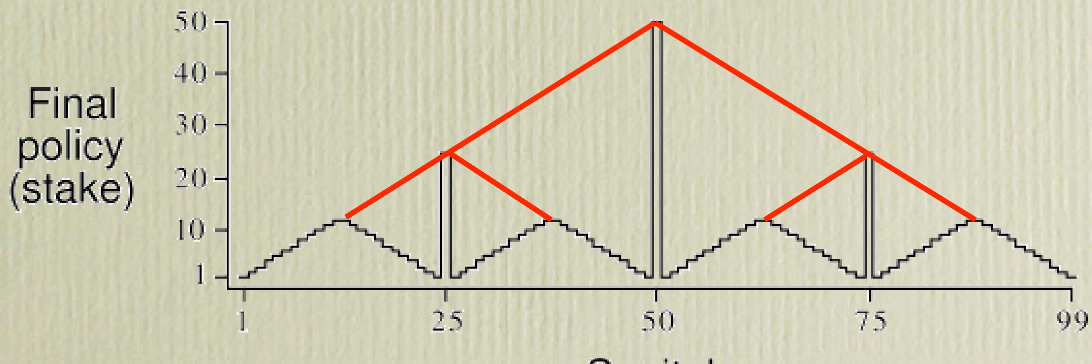
Output a deterministic policy, π, such that
  $\pi(s) = \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$

Gambler's Problem

- Gambler can repeatedly bet $ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital: $1, $2, ... , $99
- Gambler wins if his capital becomes $100; loses if it becomes $0
- Coin is unfair
  - Heads (gambler wins) with probability $p = 0.4$
- States, Actions, Rewards?

Gambler's Problem Solution





Herd Management

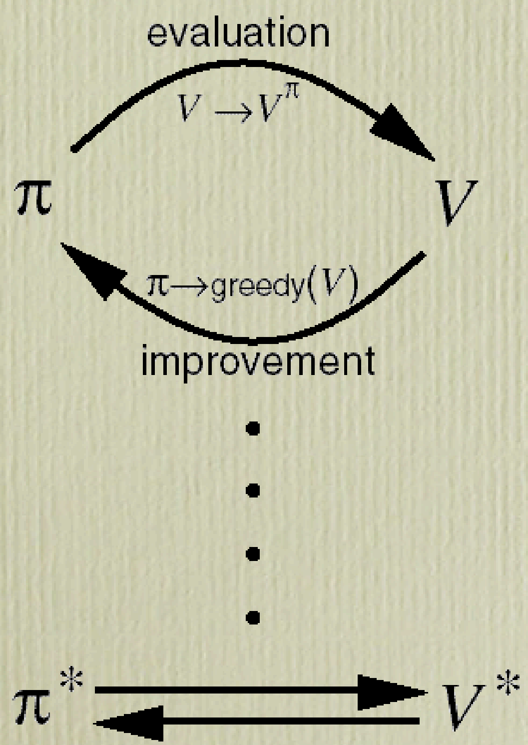
- You are a consultant to a farmer managing a herd of cows
- Herd consists of 5 kinds of cows:
  - Young
  - Milking
  - Breeding
  - Old
  - Sick
- Number of each kind is the State
- Number sold of each kind is the Action
- Cows transition from one kind to another
- Young cows can be born

Asynchronous DP

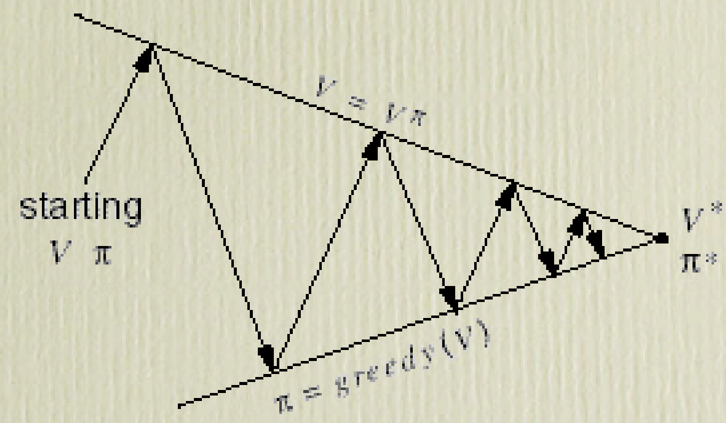
- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

Generalized Policy Iteration

Generalized Policy Iteration (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Efficiency of DP

- To find an optimal policy is polynomial in the number of states...
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called "the curse of dimensionality").
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and appropriate for parallel computation.
- It is surprisingly easy to come up with MDPs for which DP methods are not practical.

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Summary

- Policy evaluation: backups without a max
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes
- Value iteration: backups with a max
- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates